

Extension matrix representation theory of light beams and the Beauregard effect

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Abstract

It is shown that a light beam in free space is representable by an integral over a vectorial angular spectrum that is expressed in terms of an extension matrix, which describes the vectorial nature of the beam. A symmetry axis of the extension matrix is identified. When it is neither perpendicular nor parallel to the propagation axis, we arrive at such beams that show us for the first time the observable evidence of the Beauregard effect. The advanced representation theory may yield any kinds of light beam, and the uncovered Beauregard effect would play its unique roles in applications.

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Introduction The vectorial feature, or the polarization property, of a coherent light beam, which concerns its angular momentum [1, 2, 3], intensity distribution [4, 5, 6], and diffraction characteristics [7, 8], has become more and more important in diverse areas of applications, including optical tweezers and spanner [9, 10, 11], optical data storage, optical trapping and manipulation [12, 13], and dark-field imaging [14]. For a plane wave or a nearly plane wave, we have the Jones vector [15] to describe the vectorial feature of the electric or magnetic field. But for a bound beam, we have not yet had such a clear concept for its vectorial feature.

In this Letter, we advance a representation theory of free-space light beams in terms of an extension matrix acting on a two-form angular spectrum. The extension matrix in this theory plays the role of describing the vectorial feature of a bound beam. A symmetry axis of the extension matrix is identified due to the transversality property of the electromagnetic wave in free space. What is surprising is that when the symmetry axis of the extension matrix is neither perpendicular nor parallel to the propagation axis of the beam, we arrive at such beams that show us for the first time [16] the observable evidence of the Beauregard effect. By Beauregard effect we mean a novel phenomenon, surmised more than 40 years ago by Beauregard [17], that a circularly polarized beam of zero transverse wave vector can be deflected in a transverse direction. It is shown that the transverse deflection due to the Beauregard effect can be as large as the order of the beam waist under specific conditions. The uncovered Beauregard effect would open a new area of applications.

General theory We consider three-dimensional light beams that propagate in free space. Let us first write out, in the Cartesian coordinate system, the integral expression for the vectorial electric field of a beam that may propagate in positive x direction,

$$\mathbf{E}(\mathbf{x}) = \int \int_{k_y^2 + k_z^2 < k^2} \frac{dk_y dk_z}{2\pi} \mathbf{A}(k_y, k_z) \exp(i\mathbf{k} \cdot \mathbf{x}), \quad (1)$$

where the time dependence $\exp(-i\omega t)$ is implied and omitted. As we know, any element of the angular spectrum represented by its wave vector $\mathbf{k} \equiv (k_x \ k_y \ k_z)^T$ has only two independent polarization states, though its vectorial amplitude $\mathbf{A} \equiv (A_x \ A_y \ A_z)^T$ has three components, where the superscript T means transpose. Denoting respectively by s and p the two orthogonal linear polarization states, we have $\mathbf{A} = \mathbf{A}_s + \mathbf{A}_p = A_s \mathbf{s} + A_p \mathbf{p}$, where A_s and A_p are the complex amplitudes of the s and p polarization states, respectively, \mathbf{s} and \mathbf{p} are their respective unit vectors. Letting $\mathbf{s} = s_x \mathbf{e}_x + s_y \mathbf{e}_y + s_z \mathbf{e}_z$ and $\mathbf{p} = p_x \mathbf{e}_x + p_y \mathbf{e}_y + p_z \mathbf{e}_z$,

where \mathbf{e}_x , \mathbf{e}_y , and \mathbf{e}_z are the unit vectors in the directions of the Cartesian coordinates, s_j and p_j ($j = x, y, z$) are real numbers, we have

$$\mathbf{A} = \mathbf{P}\tilde{A}, \quad (2)$$

where

$$\tilde{A} = \begin{pmatrix} A_s \\ A_p \end{pmatrix} \quad (3)$$

is the two-form amplitude [16] of the angular spectrum which is, or can be regarded as, the Jones vector [15] associated with the angular spectrum, and

$$\mathbf{P} = \begin{pmatrix} s_x & p_x \\ s_y & p_y \\ s_z & p_z \end{pmatrix} \equiv (\mathbf{s} \ \mathbf{p}) \quad (4)$$

extends the two-form amplitude \tilde{A} onto the three-component vectorial amplitude \mathbf{A} and is thus referred to as the extension matrix. It is clear that it is the extension matrix rather than the two-form amplitude that describes the vectorial nature of the beam. The mutual orthogonality between the unit vectors \mathbf{s} and \mathbf{p} and the wave vector \mathbf{k} leads to

$$\begin{aligned} s_x^2 + s_y^2 + s_z^2 &= 1, \\ p_x^2 + p_y^2 + p_z^2 &= 1, \\ s_x p_x + s_y p_y + s_z p_z &= 0, \\ k_x s_x + k_y s_y + k_z s_z &= 0, \\ k_x p_x + k_y p_y + k_z p_z &= 0. \end{aligned} \quad (5)$$

We have represented a free-space light beam by Eqs. (1)-(5). But there are only the above five equations to determine the six real numbers s_j and p_j in the extension matrix. That is to say, there is one freedom to be chosen. This can be done by specifying the symmetry axis of the extension matrix. Two independent types of representation will be discussed below.

Without loss of generality, we consider the following two-form amplitude,

$$\tilde{A} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} A \equiv \tilde{l}A, \quad (6)$$

where $\tilde{l} = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ describes the polarization state of the angular spectrum and is assumed to satisfy the normalization condition $\tilde{l}^\dagger \tilde{l} = 1$, the superscript \dagger means transpose conjugate, and A is the amplitude distribution of the angular spectrum (ADAS). For the sake of simplicity, we will consider the following Gaussian distribution function [16],

$$A = \left(\frac{w_y w_z}{\pi} \right)^{1/2} \exp \left[-\frac{w_y^2}{2} (k_y - k_{y0})^2 - \frac{w_z^2}{2} k_z^2 \right], \quad (7)$$

where $w_y = w_0 / \cos \theta_0$, $w_z = w_0$, w_0 is half the width of the beam at waist, and $k_{y0} = k \sin \theta_0$. This ADAS indicates that the axis of propagation, represented by its principal wave vector $\mathbf{k}_0 = (k_{x0} \ k_{y0} \ 0)^T$, is perpendicular to the z axis, where $k_{x0} = k \cos \theta_0$. For the same reason, only beams satisfying paraxial approximation [18] $\Delta\theta = \frac{1}{kw_0} \ll 1$ will be considered, where $\Delta\theta$ is half the divergence angle of the beam. Under this condition, Eq. (1) can be rewritten as [19]

$$\mathbf{E}(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{A}(k_y, k_z) \exp(i\mathbf{k} \cdot \mathbf{x}) dk_y dk_z. \quad (8)$$

Representation of $p_z = 0$ and uniformly polarized beams When $p_z = 0$, we have for the other elements of the extension matrix that solve Eq. (5),

$$\begin{aligned} s_x &= -\frac{k_x k_z}{k(k_x^2 + k_y^2)^{1/2}}, \\ s_y &= -\frac{k_y k_z}{k(k_x^2 + k_y^2)^{1/2}}, \\ s_z &= \frac{(k_x^2 + k_y^2)^{1/2}}{k}, \\ p_x &= -\frac{k_y}{(k_x^2 + k_y^2)^{1/2}}, \\ p_y &= \frac{k_x}{(k_x^2 + k_y^2)^{1/2}}. \end{aligned}$$

Denoting $\mathbf{k}_r \equiv k_r \mathbf{e}_r = k_x \mathbf{e}_x + k_y \mathbf{e}_y$ in a cylindrical coordinate system, where $k_x = k_r \cos \theta$, $k_y = k_r \sin \theta$, and \mathbf{e}_r is the unit vector in the radial direction, it is clear that $\mathbf{p} = p_x \mathbf{e}_x + p_y \mathbf{e}_y = \mathbf{e}_\theta$ is the unit vector in the azimuthal direction. Furthermore, letting $\mathbf{s}_r = s_x \mathbf{e}_x + s_y \mathbf{e}_y$, it is found that $\mathbf{s}_r = -\frac{k_z}{k} \mathbf{e}_r$ is in the radial direction and that $s_z = \frac{k_r}{k}$. All these facts show that the symmetry axis of the extension matrix (4) is the z axis and thus is perpendicular to the axis of propagation. Because the ADAS in Eq. (7) is appreciable only in a small region in which $|k_y - k_{y0}| \sim |k_z| \leq k\Delta\theta \ll k$, k_z/k in the extension matrix can be regarded

as a small number in comparison with unity in view of the integral (8). To the zeroth-order approximation, we have for the extension matrix,

$$\mathbf{P} \approx \begin{pmatrix} 0 & -\sin \theta_0 \\ 0 & \cos \theta_0 \\ 1 & 0 \end{pmatrix}. \quad (9)$$

The electric field corresponding to extension matrix (9) is

$$\begin{aligned} \mathbf{E}_T(\mathbf{x}) &= [l_1 \mathbf{e}_z + l_2 (\mathbf{e}_y \cos \theta_0 - \mathbf{e}_x \sin \theta_0)] \\ &\times \frac{1}{2\pi} \int \int A \exp(i\mathbf{k} \cdot \mathbf{x}) dk_y dk_z, \end{aligned} \quad (10)$$

which is transverse with respect to the propagation axis \mathbf{k}_0 and is uniformly polarized. The polarization state is represented by that of the angular spectrum, \tilde{l} . Higher-order corrections will be included if higher-order approximations are considered [4]. When appropriate ADAS's are selected, the known kinds of uniformly polarized beam will be obtained, including the fundamental Gaussian beam, Hermite-Gaussian beams, Laguerre-Gaussian beams [19, 20], and the Bessel-Gaussian beams [21, 22].

Representation of $s_x = 0$, non-uniformly polarized beams, and the Beauregard effect We have in this representation the following solution to Eq. (5),

$$\begin{aligned} s_y &= -\frac{k_z}{(k_y^2 + k_z^2)^{1/2}}, \\ s_z &= \frac{k_y}{(k_y^2 + k_z^2)^{1/2}}, \\ p_x &= -\frac{(k_y^2 + k_z^2)^{1/2}}{k}, \\ p_y &= \frac{k_x k_y}{k(k_y^2 + k_z^2)^{1/2}}, \\ p_z &= \frac{k_x k_z}{k(k_y^2 + k_z^2)^{1/2}}. \end{aligned} \quad (11)$$

Denoting $\mathbf{k}_r \equiv k_r \mathbf{e}_r = k_y \mathbf{e}_y + k_z \mathbf{e}_z$ in another cylindrical coordinate system, where $k_y = k_r \cos \varphi$, and $k_z = k_r \sin \varphi$, we find that $\mathbf{s} = s_y \mathbf{e}_y + s_z \mathbf{e}_z = \mathbf{e}_\varphi$ is the unit vector in the azimuthal direction. In addition, letting $\mathbf{p}_r = p_y \mathbf{e}_y + p_z \mathbf{e}_z$, it is apparent that $\mathbf{p}_r = \frac{k_x}{k} \mathbf{e}_r$ is in the radial direction and that $p_x = -\frac{k_r}{k}$. That is to say, the extension matrix in this representation is symmetric with respect to the x axis. Therefore, θ_0 here is the angle between the symmetry axis of the extension matrix and the propagation axis. It will be

shown in the following that different values of θ_0 correspond to different beams with different polarization properties and intensity distributions. All the beams in this representation are spatially nonuniformly polarized.

The case of $\theta_0 = 0$ and axially symmetric polarization The symmetry axis of the extension matrix is parallel to the axis of propagation in this case. We use the cylindrical coordinate system by defining $\mathbf{x} = x\mathbf{e}_x + \mathbf{r}$, where $\mathbf{r} = y\mathbf{e}_y + z\mathbf{e}_z$, $y = r \cos \phi$, and $z = r \sin \phi$. The ADAS in Eq. (7) is expressed in the cylindrical coordinate system as

$$A = \frac{w_0}{\sqrt{\pi}} \exp\left(-\frac{w_0^2}{2}k_r^2\right). \quad (12)$$

Substituting Eqs. (2), (4), (6), (11), and (12) into Eq. (8) and with the help the following expansion,

$$\exp(i\rho \cos \psi) = \sum_{m=-\infty}^{\infty} i^m J_m(\rho) \exp(im\psi), \quad (13)$$

we obtain for the electric field

$$\mathbf{E}_1(\mathbf{x}) = i(l_1\mathbf{e}_\phi + l_2\mathbf{e}_r)r\chi_1(r, x) \exp(ikx) + l_2\mathbf{e}_xE_{1x}^p(\mathbf{x}), \quad (14)$$

where

$$\begin{aligned} \chi_1(r, x) &= \frac{1}{r} \int_0^\infty A_1 J_1(rk_r) k_r dk_r = \frac{\sqrt{2}w_0}{4w^3} \\ &\quad \times \exp\left(-\frac{r^2}{4w^2}\right) \left[I_0\left(\frac{r^2}{4w^2}\right) - I_1\left(\frac{r^2}{4w^2}\right) \right], \\ E_{1x}^p(\mathbf{x}) &= -\exp(ikx) \int_0^\infty \frac{k_r}{k} A_1 J_0(rk_r) k_r dk_r, \\ A_1 &= A \exp\left(-\frac{ix}{2k}k_r^2\right), \\ w(x) &= w_0 \left(1 + i\frac{x}{kw_0^2}\right)^{1/2}, \end{aligned}$$

J_m 's are the Bessel functions of the first kind, I_0 and I_1 are the modified Bessel functions of the first kind. In deriving Eq. (14), we have made (i) the paraxial approximation [19] $k_x \approx k - \frac{k_r^2}{2k}$ in the exponential factor $\exp(i\mathbf{k} \cdot \mathbf{x})$, (ii) and the zeroth-order approximation $\mathbf{p}_r = \frac{k_x}{k}\mathbf{e}_r \approx \mathbf{e}_r$ in the extension matrix.

Eq. (14) describes beams of axially symmetric polarization. The first term on the right side is the transverse component and is of the zeroth order. The second term is the longitudinal component. It is of the first order, $\sim k_r/k$, and is thus much smaller than the transverse

component [18, 23]. Neglecting the small longitudinal component, the beam is dark on the principal axis $r = 0$. Locally, it is elliptically polarized the same as the angular spectrum. Other kinds of axially symmetrically polarized beams will be obtained when appropriate ADAS's are selected, including the azimuthally and radially polarized Bessel-Gaussian and Laguerre-Gaussian beams [24, 25, 26].

The case of $|\theta_0| \leq \Delta\theta$ and the Beauregard effect If $|\theta_0|$ is not equal to zero but is very small satisfying $|\theta_0| \leq \Delta\theta$, we have approximately $w_y \approx w_0$, so that the ADAS in Eq. (7) becomes

$$A = A_0 \exp(k_{y0} w_0^2 k_y), \quad (15)$$

where

$$A_0 = \frac{w_0}{\sqrt{\pi}} \exp\left(-\frac{w_0^2}{2} k_{y0}^2\right) \exp\left(-\frac{w_0^2}{2} k_r^2\right).$$

With the help of expansion (13) and the Graf formula [27], we obtain for the beam's electric field after substituting Eqs. (2), (4), (6), (11), and (15) into Eq. (8),

$$\mathbf{E}_2(\mathbf{x}) = \mathbf{E}_{2T}(\mathbf{x}) + l_2 \mathbf{e}_x E_{2x}^p(\mathbf{x}), \quad (16)$$

where

$$\begin{aligned} \mathbf{E}_{2T}(\mathbf{x}) &= [i(l_1 \mathbf{e}_\phi + l_2 \mathbf{e}_r)r + (l_1 \mathbf{e}_z + l_2 \mathbf{e}_y)k_{y0} w_0^2] \\ &\quad \times \chi_2(\gamma, x) \exp(ikx), \\ E_{2x}^p(\mathbf{x}) &= -\exp(ikx) \int_0^\infty \frac{k_r}{k} A_2 J_0(\gamma k_r) k_r dk_r, \\ \chi_2(\gamma, x) &= \frac{1}{\gamma} \int_0^\infty A_2 J_1(\gamma k_r) k_r dk_r \\ &= \frac{\sqrt{2} w_0}{4w^3} \exp\left(-\frac{w_0^2}{2} k_{y0}^2\right) \exp\left(-\frac{\gamma^2}{4w^2}\right) \\ &\quad \times \left[I_0\left(\frac{\gamma^2}{4w^2}\right) - I_1\left(\frac{\gamma^2}{4w^2}\right) \right], \\ A_2 &= A_0 \exp\left(-\frac{ix}{2k} k_r^2\right), \\ \gamma &= (r^2 - k_{y0}^2 w_0^4 - 2ik_{y0} w_0^2 r \cos \phi)^{1/2}. \end{aligned}$$

In deriving Eq. (16), we have also made the aforementioned two approximations.

The first term on the right side of Eq. (16) is the transverse component, and the second term is the longitudinal component, which is also of the first order as before. It is interesting

to note that the transverse component consists of two parts having different polarization symmetries. One is of axially symmetric polarization and is dark on the propagation axis $r = 0$. The other is uniformly polarized. But locally both of them have the same polarization state as that of the angular spectrum, \tilde{l} . It is the interference between those two parts that yields the Beauregard effect as will be shown below.

Neglecting the small longitudinal component, the intensity distribution of the beam is found to be

$$I_2 \approx |\mathbf{E}_{2T}|^2 = (y^2 + z^2 + 2\sigma k_{y0} w_0^2 z + k_{y0}^2 w_0^4) |\chi_2|^2, \quad (17)$$

where $\sigma = i(l_1^* l_2 - l_1 l_2^*)$ is the polarization ellipticity of the angular spectrum. The first factor on the right side of Eq. (17) shows that the intensity is minimum at a point $\mathbf{r}_0 = -\mathbf{e}_z \sigma k_{y0} w_0^2$ on the z axis. The displacement of this point from the propagation axis is

$$z_0 = -\sigma k_{y0} w_0^2 = -\sigma k w_0^2 \sin \theta_0. \quad (18)$$

For angular spectra of circular polarizations $\sigma = \pm 1$, the minimum intensity is equal to zero. The single-sided depression of the beam intensity on the z axis renders the beam centroid deflected transversely in the z direction unless σ or θ_0 is equal to zero. According to Eq. (18), the deflection will be flipped to the opposite side of the propagation axis by changing either the sign of σ or the sign of θ_0 . Since the expectation value of the transverse wave vector k_z of the beam vanishes, the transverse deflection of the beam centroid in the z direction is nothing but the Beauregard effect [16]. To the best of my knowledge, this is the first time to demonstrate the observable evidence of the Beauregard effect. As an example, we show in Fig. 1 the normalized intensity distribution of the beam on plane $x = 0$, where $\sigma = 1$, and $\theta_0 = \Delta\theta = 10^{-3}$ rad. The intensity in Fig. 1 is numerically calculated directly from Eq. (16) and is well approximated by Eq. (17) as numerical calculations show. The null point is located at $z_0 \approx -w_0 = -159\lambda$ on the z axis. As a result, the beam centroid is deflected in the positive z direction by $\langle z \rangle \approx 102\lambda$ as can be obtained by using formula (24) of Ref. [16], which is as large as the order of w_0 .

The case of $|\theta_0| \geq \Delta\theta$ The electric field distribution of the beam in this case cannot be approximately expressed so clearly as have been done above. But the propagation characteristics of a paraxial beam can be studied numerically by use of the integral (8). For a

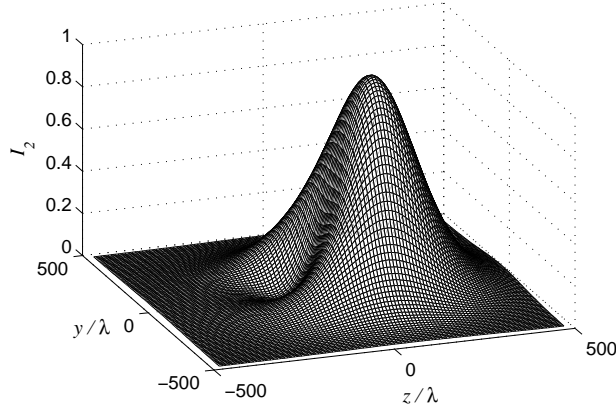


FIG. 1: Normlized intensity distribution on plane $x = 0$ for $\sigma = 1$ in representation $s_x = 0$, where $\theta_0 = \Delta\theta = 10^{-3}$ rad, the y and z coordinates are in units of the wavelength λ .

given θ_0 , the beam deflection in the transverse direction due to the Beauregard effect can be calculated by using formula (24) of Ref. [16].

In conclusion, the representation theory advanced here may produce any kinds of free-space light beam that solve the Maxwell equations. The Beauregard effect uncovered here would be closely related to the Imbert-Fedorov effect [16] and open a new area of applications. It also has vital implications to the counterpart of the quantum-mechanical matter waves [28]. In order to explore the Beauregard effect in experiments, one should be able to control separately the propagation axis of the beam, the symmetry axis of the extension matrix, as well as the polarization ellipticity of the angular spectrum.

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- [1] L. Allen, M.W. Beijersbergen, R. J. C. Spreeuw, and J.P.Woerdman, Phys. Rev. A 45, 8185 (1992).
 - [2] E. Santamato, Fortschr. Phys. 52, 1141 (2004).
 - [3] L. Marrucci, C. Manzo, and D. Paparo, Phys. Rev. Lett. 96, 163905 (2006).
 - [4] D. N. Pattanayak and G. P. Agrawal, Phys. Rev. A 22, 1159 (1980).
 - [5] L. W. Davis and G. Patsakos, Phys. Rev. A 26, 3702 (1982).

- [6] R. Dorn, S. Quabis, and G. Leuchs, Phys. Rev. Lett. 91, 233901 (2003).
- [7] R. H. Jordan and D. G. Hall, Opt. Lett. 19, 427 (1994).
- [8] P. L. Greene and D. G. Hall, J. Opt. Soc. Am. A 13, 962 (1996).
- [9] N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, Opt. Lett. 22, 52 (1997).
- [10] L. Allen, J. Opt. B: Quantum Semiclass. Opt. 4, S1 (2002).
- [11] H. Adachi, S. Akahoshi, and K. Miyakawa, Phys. Rev. A 75, 063409 (2007).
- [12] A. Ashkin, IEEE J. Sel. Top. Quantum Electron. 6, 841 (2000).
- [13] X. Xu, K. Kim, W. Jhe, and N. Kwon, Phys. Rev. A 63, 063401 (2001).
- [14] D. P. Biss, K. S. Youngworth, and T. G. Brown, Appl. Opt. 45, 470 (2006).
- [15] R. C. Jones, J. Opt. Soc. Am. 31, 488 (1941).
- [16] C.-F. Li, Phys. Rev. A 76, 013811 (2007).
- [17] O. Costa de Beauregard, Phys. Rev. 139, B1443 (1965).
- [18] M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A 11, 1365 (1975).
- [19] J. Enderlein and F. Pampaloni, J. Opt. Soc. Am. A 21, 1553 (2004).
- [20] H. Kogelnik and T. Li, Appl. Opt. 5, 1550 (1966).
- [21] F. Gori, G. Guattari, and C. Padovani, Opt. Commun. 64, 491 (1987).
- [22] V. Bagini, F. Frezza, M. Santarsiero et al, J. Mod. Opt. 43, 1155 (1996).
- [23] L. W. Davis, Phys. Rev. A 19, 1177 (1979).
- [24] D. G. Hall, Opt. Lett. 21, 9 (1996).
- [25] A. A. Tovar and G. H. Clark, J. Opt. Soc. Am. A 14, 3333 (1997).
- [26] Y. Kozawa and S. Sato, J. Opt. Soc. Am. A 24, 1793 (2007).
- [27] G. E. Andrews, Richard Askey, and R. Roy, Special Functions (Cambridge University Press, 2000), p. 215
- [28] O. Costa de Beauregard, Phys. Rev. 134, B471 (1964).